Intermediate Microeconomics Exercise Class 2

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October 8, 2022

Content

Derivative

Thanks to Yantong Xie

- Definition of derivative: $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.
- Differential and derivative: $\Delta y = A\Delta x + o(\Delta x),\ \Delta x \to 0$. We call Δy and Δx differentials. $A = \frac{dy}{dx}|_{x=x_0}$ is called differential quotient.
- Derivatives of common functions $\sin x, \cos x, \ln x, x^a, a^x \cdots$.
- Inverse function: suppose that $y = f(x)$is a bijective function then we can define $x = g(y)$. It holds that $g'(y) = \frac{1}{f'(g(y))}$.
- The derivative of implicit functions.

Example

- Derive the derivative of $f(x) = x^{x^x}$.
- Suppose that $h(x, y) = y^2 - 2xy - x^2 + 2x - 4 = 0$, prove that $y'(x) = \frac{y(x)+x-1}{y(x)-x}$.

Derivative of Composition Functions

- Suppose that $h(x) = f \circ g(x)$, then it holds that $h'(x) = g'(f(x)) \cdot f'(x)$.
- One intuitionistic explanation is $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$.

Example

Define $g(x) = f(\frac{x-1}{x+1})$, where $f(x) = \arctan x$. Derive $g'(x)$.

Partial Derivative

- Definition: fix other variates. $\partial_x f(x, y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$.
- Partial derivative of composition functions: suppose $z = f(u, v)$, $u = u(x, y)$ and $v = v(x, y)$, then it holds that

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}.$$
$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}.$$

Example

Define $f(x, y, z) = (\frac{2y}{z})^x$. Calculate partial derivatives at point $(1, 2, 1)$.

Total Differential

- Defintion: $df = \partial_x f(x_0, y_0)dx + \partial_y f(x_0, y_0)dy$.
- Gradient: $(\partial_x f, \partial_y f)$.

Lagrange-Method

- $\min f(x, y)$ *subject to* $g(x, y) = 0$. Construct $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. Then, F.O.C. is

$$\begin{cases} \partial_x F(x, y, \lambda) = 0, \\ \partial_y F(x, y, \lambda) = 0, \\ \partial_\lambda F(x, y, \lambda) = 0. \end{cases}$$

- KKT condition:

$$\begin{cases} \frac{\partial F}{\partial x} = \nabla f(x) + \lambda \nabla g(x) = 0 \\ \lambda \geq 0 \\ g(x) \leq 0 \\ \lambda g(x) = 0. \end{cases}$$

Example

- Find the difference between the height of the highest and lowest points of the curve $\begin{cases} x - y + 4z = 1 \\ 2x^2 + 4y^2 = 3 \end{cases}$ in three dimensions.

Preference, Utility and Indifference Curve

- Marginal Benefit
- Marginal Cost
- Rational Behavior: keep consuming until MB=MC

- Preference
  - Completeness
  - Reflexivity
  - Monotonicity
  - Transitivity

Preference, Utility and Indifference Curve Cont'd

- Utility
  - Ordinal Utility: monotonic transformation
  - Cardinal Utility
- Marginal Utility: law of diminishing marginal utility?
  - Quasilinear Preferences
- Marginal Rate of Substitution: $\frac{dY}{dX} = -\frac{MU_X}{MU_Y} = MRS_{XY}$.

Preference, Utility and Indifference Curve Cont'd

- Indifference Curve
  - Law of Diminishing Marginal Rate of Substitution
  - Convexity: average is better than extreme!
- Indifference Map

Example

- Cobb-Douglas Preferences
- Perfect Substitutes
- Perfect Complements

Preference, Utility and Indifference Curve Cont'd

- Economic Good
- Economic Neuter
- Economic Bad

Warm-Up

- Three Methods to Maximize Utility
  - Analysis Viewpoint
  - Geometry Viewpoint
  - Economics Viewpoint

Question 1

For lunch, Ada prefers to eat soup and bread in fixed proportions. When she eats X pints of soup, she prefers to eat $\sqrt{X}$ ounces of bread. If she has X pints of soup and more than $\sqrt{X}$ ounces of bread, she eats all the soup along with $\sqrt{X}$ ounces of bread, and throws the extra bread away. If she has X pints of soup and fewer than $\sqrt{X}$ ounces of bread (say Y ounces), she eats all the bread along with Y^2 ounces of soup and throws the extra soup away.

a) Draw Ada's indifference curves between soup and bread.

b) Assume she spends all her income on soup and bread. Plot her income-consumption curve, her Engel curve for soup, and her Engel curve for bread.

c) Derive her demand function for the two goods. [Note that demand function is a function of prices and income].

Question 2

A utility-maximizing consumer has an income $m > 0$, which he/she allocates between two goods (1 and 2). For each good, the consumer faces a constant price, p_1 and p_2, respectively. For each of the following utility function, derive the consumer's optimal demand $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.

a) $u(x_1, x_2) = \sqrt{x_1 x_2}$

b) $u(x_1, x_2) = \min\{x_1^2, x_2\}$

c) $u(x_1, x_2) = 2\sqrt{x_1} + x_2$

d) $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$

e) $u(x_1, x_2) = \begin{cases} x_1 x_2 & \text{if } x_1 \geq x_2 \\ x_1^2 & \text{if } x_1 < x_2 \end{cases}$

f) $u(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } 0 \leq x_1 \leq 20 \\ 20 + x_2 & \text{if } x_1 > 20 \end{cases}$

Question 3

Gary has two children, Kevin and Dora. Each one consumes “yummies” and nothing else. Gary loves both children equally. For example, he is equally happy when Kevin has two yummies and Dora has three, or when Kevin has three yummies and Dora has two. But he is happier when their consumption is more equal.

a) Draw Gary’s indifference curves.

b) What would they look like if he loved one child more than the other?

c) Suppose that Kevin starts out with two yummies and Dora with eight yummies, and that Gary can redistribute their yummies. Draw a “budget line” that shows his available choices and indicate his best choice by adding indifference curves.

d) How would your answer differ if Kevin started out with six yummies and Dora with four?

Question 4

You spend your monthly income on food (good 1) and books (good 2). The average price of food is constant at $p_1 = 1$. On the other hand, to encourage reading, the local bookstore runs a limited promotion: every month, the first four books you buy cost $p_2 = 1$ each, and after that books cost $p_2 = 4$ each. Your preferences are given as follows: for any two bundles $A = (x_1, x_2) > 0$ and $A' = (x_1', x_2') > 0$,

$$A \succsim A' \iff \frac{x_1}{x_1'} \geq \frac{x_2'}{x_2}.$$

a) Derive your monthly budget set and plot it in a clearly labeled graph.
b) Derive your individual demand (for both goods) as a function of income m, $x_1^*(m)$ and $x_2^*(m)$.

Thanks!